Homework 1 Solutions

Math 131B-2

- (2.5) For example purposes, let $A = \mathbb{Z}$, $B = \{.5, 1, 3, 6.2\}$, and $C = \{.5, .8, 1, 4, 9\}$.

  (a) Let $x \in A \cup (B \cup C)$. Then at least one of $x \in A$ or $x \in B \cup C$ holds. If $x \in A$, then $x \in A \cup B$. If $x \in B \cup C$, then $x \in B$ or $x \in C$ (or both), so x is in at least one of $A \cup B$ and C. In either case, $x \in (A \cup B) \cup C$. Therefore $A \cup (B \cup C) \subseteq (A \cup B) \cup C$, since every element of the first is also an element of the second. Showing the opposite inclusion is similar. We conclude the sets are equal.

  Example: $A \cup (B \cup C) = \mathbb{Z} \cup \{.5, .8, 6.2\} = (A \cup B) \cup C$.

  Now, suppose $x \in A \cap (B \cap C)$. Then by definition, $x \in A$ and $x \in B \cap C$. Since $x \in B \cap C$, we see that $x \in B$ and $x \in C$, so we now know that x is an element of each of A, B, and C. Therefore $x \in A \cap B$ and $x \in C$, so $x \in (A \cap C) \cap B$. We see that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$, since every element of the first is also an element of the second. Showing the opposite inclusion is very similar, so we conclude $A \cap (B \cap C) = (A \cap B) \cap C$).

  Example: $A \cap (B \cap C) = \mathbb{Z} \cap \{1\} = \{1\} = (\{1, 3, 4\}) \cap C = (A \cap B) \cap C$.

  (b)Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$, implying that $x \in A$ and x is in at least one of B and C. Ergo x is in at least one of $A \cap B$ and $A \cap C$, implying that $x \in (A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Conversely, if $x \in (A \cap B) \cup (A \cap C)$, then x is in at least one of $A \cap B$ and $A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x \in A \cap (B \cup C)$, and similarly if $x \in A \cap C$. Ergo $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. We conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

  Example: $A \cap (B \cup C) = \mathbb{Z} \cap \{.5, .8, 1, 3, 4, 6.2, 9\} = \{1, 3, 4, 9\} = \{1, 3\} \cup \{1, 4, 9\} = (A \cap B) \cup (A \cap C)$.

  (c)Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$, so x is an element of at least one of A and B, and $x \in A \cup C$, so x is an element of at least one of A and C. Suppose $x \in A$, then certainly $x \in A \cup (B \cap C)$. Otherwise, x must be an element of both B and C, so $x \in B \cap C$, implying that $x \in A \cup (B \cap C)$. We conclude that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Conversely, suppose $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in B \cap C$ (or both). Suppose $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Otherwise, $x \in B \cap C$, so x is an element of both B and C, implying that $x \in A \cup B$ and $x \in A \cup C$. We conclude that $x \in (A \cup C) \cap (A \cup B)$.

Ergo $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example: $(A \cup B) \cap (A \cup C) = (\mathbb{Z} \cup \{.5, 6.2\}) \cap (\mathbb{Z} \cup \{.5, .8\}) = (\mathbb{Z} \cap \{.5\}) = \mathbb{Z} \cup \{.5.1\} = A \cup (B \cap C)$.

(f) Let $x \in (A - C) \cap (B - C)$. Then x is in both $A - C$ and $B - C$. This implies that $x \in A$ and $x \in B$, but $x \notin C$, so $x \in (A \cap B) - C$. Ergo $(A - C) \cap (B - C) \subseteq (A \cap B) - C$. Conversely, let $x \in (A \cap B) - C$. Then $x \in A \cap B$ but $x \notin C$. Since $x \in A \cap B$, we see that $x \in A$ and $x \in B$, so since $x \notin C$, $x \in A - C$ and $x \in A - B$. Therefore $x \in (A - C) \cap (B - C)$, so we conclude that $(A \cap B) - C \subseteq (A - C) \cap (B - C)$, and we conclude that $(A \cap B) - C = (A - C) \cap (B - C)$.

Example: $(A - C) \cap (B - C) = (\mathbb{Z} - \{1, 3\}) \cap (\{3, 6.2\}) = \{3\} = \{1, 3\} - C = (A \cap B) - C$.

- (2.7) For example purposes, let $f(x) = x^2$, $X = [1, 2]$, $Y_1 = [0, 9)$, $Y_2 = (-1, 1)$.

  (a) Let $x \in X$. Then $f(x) \in f(X)$, so since the funtion f takes x to a point of $f(X)$, and therefore $x \in f^{-1}(f(X))$. We conclude that $X \in f^{-1}(f(X))$.

  Example: $[1, 2] \subset [-2, -1] \cup [1, 2] = f^{-1}(f([1, 2]))$.

  (c) Suppose that $x \in f^{-1}(Y_1 \cup Y_2)$. Then $f(x) \in Y_1 \cup Y_2$, so $f(x)$ is in at least one of Y_1 and Y_2. If $f(x) \in Y_1$, $x \in f^{-1}(Y_1)$, and similarly if $f(x) \in Y_2$, $x \in f^{-1}(Y_2)$. We conclude that $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$, and therefore $f^{-1}(Y_1 \cup Y_2) \subseteq f^{-1}(Y_1) \cup f^{-1}(Y_2)$. Conversely, suppose $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$. Then x is in at least one of $f^{-1}(Y_1)$ or $f^{-1}(Y_2)$. Without loss of generality, $x \in f^{-1}(Y_1)$. Then $f(x) \in Y_1 \subset Y_1 \cup Y_2$, so $x \in f^{-1}(Y_1 \cup Y_2)$. We conclude that $f^{-1}(Y_1) \cup f^{-1}(Y_2) \subseteq f^{-1}(Y_1 \cup Y_2)$. This implies the desired inequality.

  Example: $f^{-1}(Y_1 \cup Y_2) = f^{-1}((-1, 4]) = (-3, 3) = (-1, 1) \cup (-3, 3) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

  (d) Suppose $x \in f^{-1}(Y_1 \cap Y_2)$. Then $f(x) \in Y_1 \cap Y_2$, so $f(x) \in Y_1$ and $f(x) \in Y_2$. Hence $x \in f^{-1}(Y_1)$ and $x \in f^{-1}(Y_2)$, so $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Ergo $f^{-1}(Y_1 \cap Y_2) \subseteq f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Conversely, suppose $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$. Then $x \in f^{-1}(Y_1)$ and $x \in f^{-1}(Y_2)$, so $f(x) \in Y_1$ and $f(x) \in Y_2$. Hence $f(x) \in Y_1 \cap Y_2$, and therefore $x \in f^{-1}(Y_1 \cap Y_2)$. We conclude that $f^{-1}(Y_1) \cap f^{-1}(Y_2) \subseteq f^{-1}(Y_1 \cap Y_2)$. There desired inequality follows.

  Example: $f^{-1}(Y_1 \cap Y_2) = f^{-1}([0, 1)) = (-1, 1) = (-1, 1) \cap (-3, 3) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

- (2.9) We will show (a)⇒(b)⇒(d)⇒(e)⇒(c)⇒(a).

  (a)⇒(b) Let f be one-to-one on S. Observe that for any f, if $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x \in f(A)$ and $x \in f(B)$, and therefore $f(A \cap B) \subseteq f(A) \cap f(B)$. It remains to be shown that any $x \in f(A) \cap f(B)$ is in $f(A \cap B)$. If $x \in f(A) \cap f(B)$, then

$f(a) = x$ for some $a \in A$, and $f(b) = x$ for some $b \in B$. But since f is one-to-one, $a = b$, so $a \in A \cap B$, implying that $x \in f(A) \cap f(B)$. We conclude that $f(A \cap B) = f(A) \cap f(B)$.

(b)$\Rightarrow$(d) Let A and B be disjoint subsets of S. Then $\emptyset = f(A \cap B) = f(A) \cap f(B)$. Ergo $f(A)$ and $f(B)$ are also disjoint.

(d)$\Rightarrow$(e) Let A, B in S, then $A = (A \cap B) \cup (A - B)$, and $A \cap B$ and $A - B$ are disjoint. Therefore $f(A \cap B)$ and $f(A - B)$ are disjoint, and their union is $f(A)$. Hence $f(A) - f(B) = f(A - B)$.

(e)$\Rightarrow$(c) Given $A \subseteq S$, let $B = S$. Then by (d), we know $f(S - A) = f(S) - f(A)$, so there is no point $x \in S - A$ with $f(x) \in A$. We conclude that $f^{-1}[f(A)] = A$.

(c)$\Rightarrow$(a) Suppose that f is not one-to-one on S, i.e. suppose that there exists $x \neq y$ in S such that $f(x) = f(y)$. Then let $A = \{x\}$, so that $f(A) = \{f(x)\}$. But then $f^{-1}(f(A))$ contains $\{x, y\}$, and is therefore not equal to A. This contradicts (c). So if (c) holds, f must be one-to-one.

- (2.15) Observe that the set P_n of all degree $\leq n$ polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a copy of $\mathbb{Z}^n$. We can show that products of countable sets are countable, using an argument similar to the argument that $\mathbb{Q}$ is countable from class, so $\mathbb{Z}^n$ is countable. Since each such polynomial f has at most n real roots, the set of algebraic numbers A_n which are roots of degree n polynomials with integer coefficients is also countable (if we have a list of polynomials $f_1, f_2, \cdots$ we can just replace each entry in the list with its up to n roots). Then we claim that $\cup_{n=1}^{\infty} A_n$ is countable, since it is a countable union of countable sets. Hence the algebraic numbers are countable.

- (2.18) Let S be the collection of sequences whose terms are integers 0 and 1. Given any $x \in \{0, 1\}$, let us denote by x' the other element of the set. (That is, $x' = 0$ if $x = 1$ and vice versa). Suppose the set S is countable, then it is possible to make a list of sequences $s_1, s_2, s_3, \cdots$ containing every sequence in S, where each $s_n = (s_{n1}, s_{n2}, s_{n3}, \cdots)$. Consider the sequence $t = (s'_{11}, s'_{22}, s'_{33}, \cdots)$. Then $t \neq s_n$ for any n, because the nth entry of t is $s'_{nn} \neq s_{nn}$. Therefore t is a sequence with entries 0 and 1 which does not appear on our list. Ergo S cannot be countable.

- (2.19) (a) An argument similar to the proof that $\mathbb{Q}$ is countable shows that products of countable sets are countable, so in particular $\mathbb{Q}^n \subset \mathbb{R}^n$ is countable. Moreover, the set of neighbourhoods of rational radius around the points of $\mathbb{Q}^n \subset \mathbb{R}^n$ is a countable collection of neighbourhoods for each point $(q_1, \cdots q_n)$. This is another product of countable sets, and hence is countable.

(b) Suppose $\{I_\alpha : \alpha \in A\}$ is a collection of disjoint intervals of positive length on the real line. Let $[a_\alpha, b_\alpha]$ be the closures of the intervals. Because each I_α has positive length, the numbers a_α are distinct, so it suffices to make a list of real numbers containing each a_α. Moreover, since the I_α are disjoint, given any particular I_α we can always find the next interval to the right or left of I_α. In particular we can make a list of the positive a_α from left to right of the form $a_1 < a_2 < a_3 < a_4 <$ and of the negative a_α of the form $a_1' > a_2' > a_3' > \cdots$ from right to left. We combine these into a list $a_1, a_1', a_2, a_2', \cdots$. Therefore the collection $\{I_\alpha : \alpha \in A\}$ is countable.

- (3.27) (a) In $(\mathbb{R}^2, d_1)$ the set of points $\mathbf{x} = (x_1, x_2)$ which have distance less than r from $\mathbf{a} = (a_1, a_2)$ is exactly the set for which both $|x_1 - a_1| < r$ and $|x_2 - a_2| < r$. This is a square with corners $(a_1 \pm r, a_2 \pm r)$ and sides parallel to the coordinate axes.

  (b) In $(\mathbb{R}^2, d_2)$ the set of points the set of points $\mathbf{x} = (x_1, x_2)$ which have distance less than r from $\mathbf{a} = (a_1, a_2)$ is exactly the set for which $|x_1 - a_1| + |x_2 - a_2| < r$. This is a square with corners $(a_1 \pm r, a_2)$ and $(a_1, a_2 \pm r)$, and diagonals parallel to the coordinate axes.

  (c) Similar to (a).

  (d) Similar to (b).

- (3.28) It suffices to prove the desired inequalities for the squares of the metrics in question, which is somewhat more straightforward.

$$\begin{aligned} d_1(\mathbf{x}, \mathbf{y})^2 &= \max_{1 \le i \le n} |x_i - y_i|^2 \\ &= \max_{1 \le i \le n} (x_i - y_i)^2 \\ &\le (x_1 - y_1)^2 + \cdots (x_n - y_n)^2 \\ &= ||\mathbf{x} - \mathbf{y}||^{\mathbf{2}} \end{aligned}$$

Similarly, since in general $\sqrt{a_1^2 + \cdots + a_n^2} \le |a_1| + \cdots + |a_n|$, we see that

$$\begin{aligned} ||\mathbf{x}, \mathbf{y}||^2 &= (x_1 - y_1)^2 + \cdots (x_n - y_n)^2 \\ &\le (x_1 - y_1)^2 + \cdots (x_n - y_n)^2 + \sum_{1 \le i < j \le n} 2|x_i - y_i||x_j - y_j| \\ &= (|x_1 - y_1| + \cdots |x_n - y_n|)^2 \\ &= d_2(\mathbf{x}, \mathbf{y})^2 \end{aligned}$$

This shows that $d_1(\mathbf{x}, \mathbf{y}) \le ||\mathbf{x}, \mathbf{y}|| \le d_2(\mathbf{x}, \mathbf{y})$. For the other inequality, we observe that

$$\begin{aligned} n||\mathbf{x}-\mathbf{y}|| &= n[(x_1-y_1)^2+\cdots+(x_n-y_n)^2] \\ &\le n[n\max_{1\le i\le n}(x_i-y_i)^2] \\ &= n^2 d_1(\mathbf{x},\mathbf{y})^2. \end{aligned}$$

For the last part of the inequality, we need to observe that in general for nonnegative $a_1,\cdots,a_n$, we have $n(a_1^2+\cdots a_n^2)\ge(a_1+\cdots a_n)^2$. Cancelling terms, this is the same as $(n+1)(a_1^2+\cdots a_n^2)\ge\sum_{1\le i<jn}2a_ia_j$. To see why this is true, notice that for any positive a and b, $(a-b)^2\ge 0$, so $a^2-2ab+b^2\ge 0$, implying that $a^2+b^2\ge 2ab$. The general statement follows from repeated applications of this fact. Ergo we compute that

$$\begin{aligned} n||\mathbf{x}-\mathbf{y}|| &= n[(x_1-y_1)^2+\cdots+(x_n-y_n)^2] \\ &\ge [(x_1-y_1)^2+\cdots(x_n-y_n)^2+\sum_{1\le i<j\le n}2|x_i-y_i||x_j-y_j|] \quad = d_2(\mathbf{x},\mathbf{y})^2 \end{aligned}$$

We conclude that $d_2(\mathbf{x},\mathbf{y})\le\sqrt{n}||\mathbf{x}-\mathbf{y}||\le nd_1(\mathbf{x},\mathbf{y})$.

- (3.29) Let (M,d) be a metric space, and

$$d'(x,y)=\frac{d(x,y)}{1+d(x,y)}.$$

We check that d' is a metric.

  - For any $x\in M$, $d'(x,x)=\frac{d(x,x)}{1+d(x,x)}=\frac{0}{1}=0$.
  - (Positivity) For any $x,y\in M$ such that $x\neq y$, $d(x,y)>0$ and $1+d(x,y)>0$, so $d'(x,y)>0$.
  - (Symmetry) For any $x,y\in M$, $d(x,y)=d(y,x)$ and $1+d(x,y)=1+d(y,x)$, so $d'(x,y)=d'(y,x)$.
  - (Triangle Inequality) Let $x,y,z\in M$. Then

$$\begin{aligned} d'(x,y)-d'(z,y) &= \frac{d(x,y)}{1+d(x,y)}-\frac{d(z,y)}{1+d(z,y)} \\ &= \frac{d(x,y)(1+d(z,y))-d(z,y)(1+d(x,y))}{(1+d(x,y))(1+d(z,y))} \\ &= \frac{d(x,y)-d(z,y)}{1+d(x,y)+d(z,y)+d(x,y)d(z,y)} \\ &\le \frac{d(x,z)}{1+d(x,y)+d(z,y)+d(x,y)d(z,y)} \\ &\le \frac{d(x,z)}{1+d(x,z)} \\ &= d'(x,z) \end{aligned}$$

Here the last steps use the triangle inequality twice: we know that $d(x, y) - d(z, y) \leq d(x, z)$, and moreover we know that $d(x, y) + d(z, y) + d(x, y)d(z, y) \geq d(x, y) + d(z, y) \geq d(x, z) \geq 0$. Ergo $d'(x, y) \leq d'(x, z) + d'(z, y)$, which is the triangle inequality. (Unsurprisingly, there are many different ways to work through this algebra.)